Vol. 2 No. 1 page 35-40
DOI: 10.14456/randk.2016.4

RESEARCH & KNOWLEDGE

Research Article

Quantum theory of matter in bulk: Modern treatment

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(Received 23 March 2016; accepted 24 May 2016)

Abstract - A systematic mathematical modern presentation is given, providing in a direct way the underlying technical details, to show how quantum theory, with the Pauli exclusion principle, has, over the years, solved the problem of why matter in bulk is stable and occupies so large a volume.

Keywords: Quantum theory of matter in bulk, fundamental role of the Pauli exclusion principle against collapse, large extension of matter

1. Introduction

One of the greatest problems that quantum mechanics has addressed over the years is the problem of why matter in bulk is stable, after realizing that classical theory fails to do so. That is, why matter, around us in our world, does not simply collapse. This paper deals in a *direct* mathematically rigorous way the fundamental role that quantum theory has played, over the years, in one of the most important problems of theoretical physics. If one prepares a list of the most important problems in quantum theory addressed over the years, this subject will undoubtedly be on it. The purpose of this communication is to spell out in a modern, comprehensive and rigorous way by invoking the Pauli exclusion principle, the underlying mathematics involved in the quantum mechanics that establishes matter in bulk is stable, and, for completeness, elaborate, rigorously as well, on the unusually large volume matter occupies. The latter was clearly emphasized in *words* by Ehrenfest to Pauli in 1931 on the occasion of the Lorentz medal (Ehrenfest, 1595) to this effect: "We take a piece of metal, or a stone. When we think about it, we are astonished that this quantity of matter should occupy so large a volume." He went on by stating that the Pauli exclusion principle is the reason: "Answer: only the Pauli Principle, no two electrons in the same state." On the other hand, if the Pauli exclusion principle is not invoked, it is interesting to quote Dyson (Dyson, 1967) who states: "[such] matter in bulk would collapse into a condensed high-density phase. The assembly of any two macroscopic objects would release energy comparable to that of an atomic bomb. Matter without the exclusion principle is unstable." In the translated version of the book by Tomonaga on spin (Tomonaga, 1997), one reads in the Preface: "The existence of spin, and the statistics associated with it, is the most subtle and ingenious design of Nature - without it the whole universe would collapse."

The drastic difference between matter, with the exclusion principle, and "bosonic matter," i.e., for which the Pauli exclusion principle is not invoked, with Coulomb interactions, is that the ground-state energy for the latter, $E_N \sim -N^{\alpha}$, with $\alpha>1$, where $(N+N)$ denotes the number of the negatively charged particles plus an equal number of positively charged particles. This behavior for "bosonic matter" is unlike that of matter, with the exclusion principle, for which $\alpha=1$ (see Dyson, 1967; Dyson and Lenard, 1967; Lenard and Dyson, 1968; Lieb and Thirring, 1975; Lieb, 1979; Manoukian and Muthaporn 2002; Manoukian and Muthaporn, 2003; Muthaporn and Manoukian, 2004; Manoukian and Sirininlakul, 2004; Manoukian and Muthaporn, 2003; Muthaporn and Manoukion, 2004). A power law behavior with $\alpha>1$, implies instability, as the formation of a single system consisting of $(2N+2N)$ particles is favored over two separate systems brought together each consisting of $(N+N)$ particles, and the energy released upon collapse of the two systems into one, being proportional to $[(2N)^{\alpha}-2(N)^{\alpha}]$, will be overwhelmingly large for realistic large N, e.g., $N\sim 10^{23}$. The instability of "bosonic matter" is not a characteristic of the dimensionalityof space (Muthaporn and Manoukian, 2004). We have been particularly interested in recent years on the density limit of matter with (Manoukian and Sirininlakul, 2005) and without (Manoukian *et al.,* 2006) the exclusion principle, and the size of such matter in bulk as more and more matter is put together from the point of view mentioned above by Ehrenfest. For completeness, we thus also elaborate rigorously on the large extent (Manoukian, 2013) of matter and its intimate connection with the exclusion principle. Our findings are summarized in the concluding section, which also pin points the strategy of attack and how the explicit statements of the extension of matter are extracted from the theory. The underlying *technical* details, not just in words, are spelled out and

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given right here in the bulk of the paper that lead to our explicit conclusions.

The Hamiltonian of consideration in this work is defined by the well known expression

$$H=\sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m}+\sum_{i<j}^{N}\frac{e^2}{|\mathbf{x}_i-\mathbf{x}_j|}-\sum_{i=1}^{N}\sum_{j=1}^{k}\frac{Z_j e^2}{|\mathbf{x}_i-\mathbf{R}_j|}+\sum_{i<j}^{k}\frac{Z_i Z_j e^2}{|\mathbf{R}_i-\mathbf{R}_j|}, \quad (1)$$

where $Z_i|e|$ denotes the charge of a j^{th} positively charged particle, $\mathbf{x}_i$, $\mathbf{R}_j$, corresponding, respectively, to the positions of the negatively and the positively charged particles, and m denotes the mass of the negatively charged particles. We also consider neutral matter, that is, $\sum_{j=1}^{k} Z_j = N$. The solution of the stability problem rests on a basic ine quality derived by the legendary Julian Schwinger (Schwinger, 1961) given in §2, followed by a non-binding theorem attributed to Edward Teller (Teller, 1962 ; Lieb and Thirring, 1975) in §3. The N power law of the ground-state energy of matter and its stability is established in §4. The large volume aspect of matter is elaborated upon next, followed by our conclusions in §5.

2. The Schwinger Bound

Given a Hamiltonian

$$h=\frac{\mathbf{p}^2}{2m}-f(\mathbf{x}), \qquad f(\mathbf{x})\geq 0 \quad (2)$$

in three dimensional space, then the number of eigenvalues less than a parameter $-\xi$, where $\xi>0$, denoted by $N[h,-\xi]$, satisfies the inequality

$$N[h,-\xi]\leq\left(\frac{m}{2\hbar^2}\right)^{3/2}\frac{1}{\pi\sqrt{\xi}}\int d^3\mathbf{x}\, f^2(\mathbf{x}). \quad (3)$$

If we choose

$$-\xi=-\frac{1+\epsilon}{\pi^2}\left(\frac{m}{2\hbar^2}\right)^3\left(\int d^3\mathbf{x}\, f^2(\mathbf{x})\right)^2, \quad (4)$$

then $N[h,-\xi]<1$, that is $N[h,-\xi]=0$, and the spectrum is empty below the value on the right-hand side of (4). Hence the right-hand side of (4) gives the following *lower bound* to the spectrum of h

$$h\geq-\frac{1+\epsilon}{\pi^2}\left(\frac{m}{2\hbar^2}\right)^3\left(\int d^3\mathbf{x}\, f^2(\mathbf{x})\right)^2, \quad (5)$$

for any $\epsilon>0$.

Another useful formula for obtaining a lower bound to the Hamiltonian in (2) is obtained from (3) by integrating the latter over ξ. This will give us an upper bound to the sum of the negative eigenvalues of the Hamiltonian $h=\left[\mathbf{p}^2/2m-f(\mathbf{x})\right]$ as in (2). To this end, we use the identity

$$N[h_0-f,-\xi;\xi>0]=N[h_0-(f-\frac{\xi}{2});-\frac{\xi}{2};0<\xi\leq 2f(\mathbf{x})],$$
$$h_0=\frac{\mathbf{p}^2}{2m}. \quad (6)$$

That is,

$$\int_0^\infty d\xi\, N[h_0-f,-\xi]\leq\left(\frac{m}{2\hbar^2}\right)^{3/2}\frac{\sqrt{2}}{\pi}\int d^3\mathbf{x}\int_0^{2\upsilon(\mathbf{x})}\frac{d\xi}{\sqrt{\xi}}\left(f(\mathbf{x})-\frac{\xi}{2}\right)^2, \quad (7)$$

which leads to

$$\int_0^\infty d\xi N[h_0-f,-\xi]\leq\frac{4}{15\pi}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int d^3\mathbf{x}\left(f(\mathbf{x})\right)^{5/2}, \quad (8)$$

referred to as a Lieb-Thirring bound (Lieb and Thirring, 1975), providing an upper bound for the negative of the sum of the negative eigenvalues (if any), counting degeneracy, of a Hamiltonian h, such as the one in (2). Since the ground-state energy cannot be less than the sum of the negative eigenvalues, this equation provides a lower bound for the ground-state energy with

$$h\geq-\frac{4}{15\pi}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int d^3\mathbf{x}(f(x))^{(5/2)}. \quad (9)$$

3. A Non-Binding Theorem

We introduce the functional

$$F[\varrho;Z_1,...,Z_k,\mathbf{R}_1,...,\mathbf{R}_k]$$
$$=(3\pi^2)^{5/3}\frac{\hbar^2}{10\pi^2 m\beta}\int d^3\mathbf{x}\,\varrho^{5/3}(\mathbf{x})-\sum_{j=1}^{k}Z_j\,e^2\int d^3\mathbf{x}\frac{\varrho(\mathbf{x})}{|\mathbf{x}-\mathbf{R}_j|}$$
$$+\frac{e^2}{2}\int d^3\mathbf{x}\,d^3\mathbf{x}'\varrho(\mathbf{x})\frac{1}{|\mathbf{x}-\mathbf{x}'|}\varrho(\mathbf{x}')+\sum_{1\leq i<j\leq k}\frac{Z_i Z_j e^2}{|\mathbf{R}_i-\mathbf{R}_j|}. \quad (10)$$

Here $\varrho(\mathbf{x})$ is an arbitrary positive function, and $\beta>0$ is an arbitrary dimensio nless parameter. Also, $Z_j|e|$ denotes the charge of a j^{th} positively charged particle and $\mathbf{R}_j$, corresponds to the positions of these positively charged particles - the nuclei. The functional in (10) is minimized for ϱ and taken to be ϱ_0 satisfying the equation

$$(3\pi^2)^{2/3}\frac{\hbar^2}{2m\beta}\varrho_0^{2/3}(\mathbf{x};k)=$$
$$\sum_{i=1}^{k}\frac{Z_i\,e^2}{|\mathbf{x}-\mathbf{R}_i|}-e^2\int d^3\mathbf{x}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\varrho_0(\mathbf{x}';k). \quad (11)$$

as obtained by the functional differentiation of (10) with respect to ϱ; and by setting the result equal to zero as done in Lagrangian mechanics. That is,

$$F[\varrho; Z_1, ..., Z_k, \mathbf{R}_1, ..., \mathbf{R}_k] \geq F[\varrho_0; Z_1, ..., Z_k, \mathbf{R}_1, ..., \mathbf{R}_k]. \quad (12)$$

In particular, let $\varrho_{\mathrm{TF}}^{(i)}$ satisfy the equation

$$(3\pi^2)^{2/3} \frac{\hbar^2}{2m\beta} \left(\varrho_{\mathrm{TF}}^{(i)}(\mathbf{x})\right)^{2/3}(\mathbf{x}) = \frac{Z_i\, \mathrm{e}^2}{|\mathbf{x}-\mathbf{R}_i|} - \mathrm{e}^2 \int \mathrm{d}^3\mathbf{x}' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\, \varrho_{\mathrm{TF}}^{(i)}(\mathbf{x}'), \quad (13)$$

where $\varrho_{\mathrm{TF}}^{(i)}$ is the so-called Thomas-Fermi density with $m \to m\beta,\ Z \to Z_i$, and from (12), we have (Lieb and Thirring, 1975; Teller, 1962; Wightman et. al, 1991)

$$F[\varrho_0; Z_1, ..., Z_k, \mathbf{R}_1, ..., \mathbf{R}_k] \geq \sum_{i=1}^{k} F[\varrho_{\mathrm{TF}}^{(i)}; Z_i, \mathbf{R}_i], \quad (14)$$

and finally

$$F[\varrho; Z_1, ..., Z_k, \mathbf{R}_1, ..., \mathbf{R}_k] \geq \beta E_{\mathrm{TF}}(1) \sum_{i=1}^{k} Z_i^{7/3}, \quad (15)$$

where $E_{\mathrm{TF}}(Z)$ is the Thomas-Fermi energy for atoms, and numerically

$$E_{\mathrm{TF}}(1) \simeq -1.5375\left(\frac{m\mathrm{e}^4}{2\hbar^2}\right). \quad (16)$$

This inequality states that a system identified by the parameters $[Z_1, ..., Z_k, \mathbf{R}_1, ..., \mathbf{R}_k]$ cannot have an (optimized) energy functional (10) less than the sum of the (optimized) energy functionals of any two subsystems identified by the parameters $[Z_1, ...Z_\ell, \mathbf{R}_1, ...\mathbf{R}_\ell]$, $[Z_{\ell+1}, ..., Z_k, \mathbf{R}_{\ell+1}, ..., \mathbf{R}_k]$, for $\ell < k$. Due to this, the theorem embodied in the inequality is referred to as a no binding theorem.

4. The N Power Law of the Ground-State Energy of Matter and Stability

In detail (15) reads

$$(3\pi^2)^{5/3} \frac{\hbar^2}{10\pi^2\, m\, \beta} \int \mathrm{d}^3\mathbf{x}\, \varrho^{5/3}(\mathbf{x}) - \sum_{j=1}^{k} Z_j\, \mathrm{e}^2 \int \mathrm{d}^3\mathbf{x} \frac{\varrho(\mathbf{x})}{|\mathbf{x}-\mathbf{R}_j|} + \frac{\mathrm{e}^2}{2} \int \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{x}'\, \varrho(\mathbf{x}) \frac{1}{|\mathbf{x}-\mathbf{x}'|} \varrho(\mathbf{x}') + \sum_{1 \leq i < j \leq k} \frac{Z_i\, Z_j \mathrm{e}^2}{|\mathbf{R}_i - \mathbf{R}_j|} \ \geq\ \beta\, E_{\mathrm{TF}}(1) \sum_{j=1}^{k} Z_j^{7/3}. \quad (17)$$

where we recall that ϱ is an arbitrary positive function. From the above inequality, we may find a lower bound to the (repulsive) potential interaction part between the nuclei to be

$$\sum_{1 \leq i < j \leq k} \frac{Z_i\, Z_j \mathrm{e}^2}{|\mathbf{R}_i - \mathbf{R}_j|} \geq \sum_{j=1}^{k} Z_j\, \mathrm{e}^2 \int \mathrm{d}^3\mathbf{x} \frac{\varrho(\mathbf{x})}{|\mathbf{x}-\mathbf{R}_j|} - \frac{\mathrm{e}^2}{2} \int \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{x}'\, \varrho(\mathbf{x}) \frac{1}{|\mathbf{x}-\mathbf{x}'|} \varrho(\mathbf{x}') - (3\pi^2)^{5/3} \frac{\hbar^2}{10\pi^2\, m\, \beta} \int \mathrm{d}^3\mathbf{x}\, \varrho^{5/3}(\mathbf{x}) + \beta\, E_{\mathrm{TF}}(1) \sum_{j=1}^{k} Z_j^{7/3}. \quad (18)$$

This inequality, in turn, allows us to find a lower bound to the (repulsive) potential interaction part between the electrons, by making the substitutions: $k \to N$, $Z_j \to 1$, $\mathbf{R}_j \to \mathbf{x}_j$, for $j = 1, ..., N$:

$$\sum_{1 \leq i < j \leq N} \frac{\mathrm{e}^2}{|\mathbf{x}_i - \mathbf{x}_j|} \geq \sum_{j=1}^{N} \mathrm{e}^2 \int \mathrm{d}^3\mathbf{x} \frac{\varrho(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_j|} - \frac{\mathrm{e}^2}{2} \int \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{x}'\, \varrho(\mathbf{x}) \frac{1}{|\mathbf{x}-\mathbf{x}'|} \varrho(\mathbf{x}') - (3\pi^2)^{5/3} \frac{\hbar^2}{10\pi^2\, m\, \beta} \int \mathrm{d}^3\mathbf{x}\, \varrho^{5/3}(\mathbf{x}) + \beta\, E_{\mathrm{TF}}(1)\, N. \quad (19)$$

Since ϱ is an arbitrary positive function, we may take it to denote the electron density

$$\varrho(\mathbf{x}) = N \sum_{\sigma_1, \ldots, \sigma_N} \int \mathrm{d}^3\mathbf{x}_2 \ldots \mathrm{d}^3\mathbf{x}_N |\Psi(\mathbf{x}\sigma_1, \mathbf{x}_2\sigma_2, ..., \mathbf{x}_N\sigma_N)|^2, \quad (20)$$

where $\Psi(\mathbf{x}\sigma_1, \mathbf{x}_2\sigma_2, ..., \mathbf{x}_N\sigma_N)$ is a normalized wave function, anti-symmetric under the interchange of any pair $(\mathbf{x}_i\sigma_i) \leftrightarrow (\mathbf{x}_j\sigma_j)$, and the sums are over spins. The total number of particles is obtained by integrating over the number density $\varrho(\mathbf{x})$

$$\int \mathrm{d}^3\mathbf{x}\, \varrho(\mathbf{x}) = N. \quad (21)$$

We also need to derive a lower bound to the expectation value of the kinetic energy:

$$T = \sum_{\sigma_1, \ldots, \sigma_N} \int \mathrm{d}^3\mathbf{x}_1 \ldots \mathrm{d}^3\mathbf{x}_N \Psi^*(\mathbf{x}_1\sigma_1, \mathbf{x}_2\sigma_2, ..., \mathbf{x}_N\sigma_N) \times \Big(\sum_{i=1}^{N} \frac{-\hbar^2\nabla_i^2}{2m}\Big) \Psi(\mathbf{x}_1\sigma_1, \mathbf{x}_2\sigma_2, ..., \mathbf{x}_N\sigma_N). \quad (22)$$

To this end, we use the Schwinger bound in (9) and introduce, in the process, a hypothetical Hamiltonian

$$\sum_{i=1}^{N} \Big(\frac{-\hbar^2\nabla_i^2}{2m} - f(\mathbf{x}_i) \Big), \quad (23)$$

where

$$f(\mathbf{x}) = \frac{5}{3} \frac{\varrho^{2/3}(\mathbf{x})}{\int \mathrm{d}^3\mathbf{x}\, \varrho^{5/3}(\mathbf{x})} T, \qquad T = \langle \Psi | \sum_{j=1}^{N} \frac{\mathbf{p}_j^2}{2m} | \Psi \rangle. \quad (24)$$

It is easily verified that

$$\langle \Psi | \sum_{j=1}^{N} f(\mathbf{x}_j) | \Psi \rangle = \frac{5}{3} T. \quad (25)$$

Allowing multiplicity and spin degeneracy, we can put the N fermions in the lowest energy levels of the hypothetical Hamiltonian $[\sum_{j=1}^{N} (\mathbf{p}_j^2/2m - f(\mathbf{x}_j))]$, in conformity with the Pauli exclusion principle, if $N \leq$ the number of such levels. If N is larger than this number of levels, the remaining free fermions may be chosen to have arbitrary small $(\to 0)$ kinetic energies, and be infinitely separated, to define the lowest energy of this Hamiltonian. Hence in all cases, this Hamiltonian is bounded below by 2, for allowing spin orientations, times the sum of the negative energy levels of the Hamiltonian, $\left(\mathbf{p}^2/2m - f(\mathbf{x};)\right)$, allowing in the sum for multiplicity but not for spin degeneracy. Hence we obtain the bound

$$\langle\Psi|\sum_{j=1}^{N}\Big(\frac{\mathbf{p}_j^2}{2m}-f(\mathbf{x}_j)\Big)|\Psi\rangle \geq -2\frac{4}{15\pi}\Big(\frac{2m}{\hbar^2}\Big)^{3/2}\int \mathrm{d}^3\mathbf{x}(f(\mathbf{x}))^{5/2}. \qquad (26)$$

From (25), (26),

$$-\frac{2}{3}T \geq -2\frac{4}{15\pi}\Big(\frac{2m}{\hbar^2}\Big)^{3/2}\Big(\frac{5}{3}\Big)^{5/2}T^{5/2}\Big(\int \mathrm{d}^3\mathbf{x}\,\varrho^{5/3}(\mathbf{x})\Big)^{-3/2}, \qquad (27)$$

leading to

$$\frac{3}{5}\Big(\frac{3\pi}{4}\Big)^{2/3}\frac{\hbar^2}{2m}\int \mathrm{d}^3\mathbf{x}\,\varrho^{5/3}(\mathbf{x}) \leq T. \qquad (28)$$

Now we have all the ingredients to obtain a lower bound of the Hamiltonian in (1). To this end,

$$\sum_{i=1}^{N}\sum_{j=1}^{k} Z_j\,\mathrm{e}^2\Big\langle\Psi\Big|\frac{1}{|\mathbf{x}_i-\mathbf{R}_j|}\Big|\Psi\Big\rangle = \sum_{j=1}^{k} Z_j\,\mathrm{e}^2\int \mathrm{d}^3\mathbf{x}\,\frac{1}{|\mathbf{x}-\mathbf{R}_j|}\varrho(\mathbf{x}). \qquad (29)$$

$$\sum_{i=1}^{N}\mathrm{e}^2\int \mathrm{d}^3\mathbf{x}\,\varrho(\mathbf{x})\Big\langle\Psi\Big|\frac{1}{|\mathbf{x}-\mathbf{x}_i|}\Big|\Psi\Big\rangle = \mathrm{e}^2\int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}'\varrho(\mathbf{x})\frac{1}{|\mathbf{x}-\mathbf{x}'|}\varrho(\mathbf{x}'), \qquad (30)$$

and hence from (19)

$$\sum_{1\leq i<j\leq N}\mathrm{e}^2\Big\langle\Psi\Big|\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|}\Big|\Psi\Big\rangle \geq \frac{\mathrm{e}^2}{2}\int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}'\varrho(\mathbf{x})\frac{1}{|\mathbf{x}-\mathbf{x}'|}\varrho(\mathbf{x}') - (3\pi^2)^{5/3}\frac{\hbar^2}{10\pi^2 m\beta}\int \mathrm{d}^3\mathbf{x}\,\varrho^{5/3}(\mathbf{x}) + \beta N E_{\mathrm{TF}}(1). \qquad (31)$$

Moreover from (28) - (31), (1), we have

$$\langle\Psi|H|\Psi\rangle \geq (3\pi^2)^{5/3}\frac{\hbar^2}{10\pi^2 m\beta'}\int \mathrm{d}^3\mathbf{x}\,\varrho^{5/3}(\mathbf{x}) - \sum_{j=1}^{k}Z_j\,\mathrm{e}^2\int \mathrm{d}^3\mathbf{x}\,\frac{1}{|\mathbf{x}-\mathbf{R}_j|}\varrho(\mathbf{x}) + \frac{\mathrm{e}^2}{2}\int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}'\varrho(\mathbf{x})\frac{1}{|\mathbf{x}-\mathbf{x}'|}\varrho(\mathbf{x}') + \sum_{1\leq i<j\leq k}\frac{Z_iZ_j\mathrm{e}^2}{|\mathbf{R}_i-\mathbf{R}_j|} + \beta N E_{\mathrm{TF}}(1), \qquad (32)$$

where

$$\frac{1}{\beta'} = \frac{\frac{3}{5}\Big(\frac{3\pi}{4}\Big)^{2/3} - \frac{(3\pi^2)^{5/3}}{5\pi^2}\frac{1}{\beta}}{\frac{(3\pi^2)^{5/3}}{5\pi^2}} = \Big(\frac{1}{4\pi}\Big)^{2/3} - \frac{1}{\beta}. \qquad (33)$$

For a positive β', we must choose $\beta > (4\pi)^{2/3}$. The sum of the first four terms on the right-hand side of the inequality in (32) coincides with the expression on the left-hand side of the inequality (17) with β in the latter simply replaced by β'. Hence

$$\langle\Psi|H|\Psi\rangle \geq \beta' E_{\mathrm{TF}}(1)\sum_{j=1}^{k} Z_j^{7/3} + \beta N E_{\mathrm{TF}}(1). \qquad (34)$$

Optimizing over β leads to the Lieb-Thirring bound (Lieb and Thirring, 1975)

$$\langle\Psi|H|\Psi\rangle \geq E_{\mathrm{TF}}(1)(4\pi)^{2/3}N\Big[1+\Big(\sum_{j=1}^{k}\frac{Z_j^{7/3}}{N}\Big)^{1/2}\Big]^2, \qquad (35)$$

where E_{TF} is given in (16). Setting $Z = \max_j Z_j$, we obtain

$$\langle\Psi|H|\Psi\rangle \geq -8.3104\Big(\frac{m\mathrm{e}^4}{2\hbar^2}\Big)N\Big[1+Z^{2/3}\Big]^2. \qquad (36)$$

The numerical value 8.3104 may be further reduced (Hundertmark, 2000; Dolbeault *et al*., 2008), but this will not be important in the subsequent analysis (see also (Federbush, 1975; Graf, 1997). The left-hand side of the inequality in (36) provides a lower bound to the spectrum.

An upper bound to the ground-energy is also readily derived by noting that any trial wave function *cannot* give a lower bound to the ground-state energy, otherwise this would contradict the very definition of the ground-state energy. A trial wave function may be chosen to obtain (Manoukian, 2013; p.779) the following upper bound to the ground-state energy E_N:

$$E_N \leq -0.0450\Big(\frac{m\mathrm{e}^4}{2\hbar^2}\Big)N, \qquad (37)$$

thus establishing the N power law behavior of the ground-state energy $E_N \sim -N$, with a *finite* (negative) numerical coefficient, as discussed in the Introduction.

Note that the negative spectrum of the Hamiltonian in (1) is not empty for matter. Envisage a situation where we have infinitely separated N clusters: k hydrogenic atoms in their ground states, of nuclear charges $Z_1|\mathrm{e}|, \ldots, Z_k|\mathrm{e}|$, each having one negatively charged particle, and there are also $(N-k)$ free negatively charged particles with vanishingly small kinetic energies. The ground-state of such a system is $-\sum_{i=1}^{k} Z_i^2 m\mathrm{e}^4/2\hbar^2$. Let $|\varphi(m)\rangle$ denote a normalized strictly negative energy state of matter. That is,

$$-\varepsilon_N[m] \leq \langle\varphi(m)|H|\varphi(m)\rangle < 0, \qquad (38)$$

where $-\varepsilon_N[m] = E_N < 0$ denotes the lower end of the spectrum, and we have emphasized its dependence on the mass m. By definition of the ground-state, the state $|\varphi(m/2)\rangle$ cannot lead for $\langle\varphi(m/2)|H|\varphi(m/2)\rangle$ to a numerical value lower than $-\varepsilon_N[m]$ for the same Hamiltonian with mass m. That is,

$$-\varepsilon_N[m] \leq \langle\varphi(m/2)|H|\varphi(m/2)\rangle, \qquad (39)$$

where we note that the interaction part V in the Hamiltonian in (1) is independent of the mass scale m. Accordingly, we may rewrite the above equation in detail as

$$-\varepsilon_N[m] \leq \langle \varphi(m/2) | \Big[\sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + V\Big] | \varphi(m/2) \rangle. \tag{40}$$

This equation, in turn implies that for $m \to 2m$,

$$-\varepsilon_N[2m] \leq \langle \varphi(m) | \Big[\sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{4m} + V\Big] | \varphi(m) \rangle. \tag{41}$$

Upon simplifying

$$\sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + V = \Big[\sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{4m} + V\Big] + \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{4m}. \tag{42}$$

Eqs. (40), (41) imply that

$$\langle \varphi(m) | \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} | \varphi(m) \rangle \leq 2\varepsilon_N[2m], \tag{43}$$

for all states $|\varphi(m)\rangle$ for which (38) is true.

From the bounds to the spectra in (36), together with the lower bound of the expectation value of the kinetic energy part in (28), we then have the following bounds

$$\frac{3}{5}\Big(\frac{3\pi}{4}\Big)^{2/3} \frac{\hbar^2}{2m} \int \mathrm{d}^3\mathbf{x}\, \varrho^{5/3}(\mathbf{x}) < \Big\langle \Psi \Big| \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} \Big| \Psi \Big\rangle < 16.63\Big(\frac{me^4}{\hbar^2}\Big) N \big[1 + Z^{2/3}\big]^2. \tag{44}$$

This, in turn, gives the following key bound for the integral of some power of the particle *density* $\varrho(\mathbf{x})$:

$$\int \mathrm{d}^3\mathbf{x}\, \varrho^{5/3}(\mathbf{x}) < 32\, \frac{m^2 \mathrm{e}^4}{\hbar^4} N\, [1 + Z^{2/3}]^2. \tag{45}$$

Now let $\mathbf{x}$ denote the position of an electron relative, for example, to the center of the mass of the nuclei, recalling that the Pauli exclusion was invoked in deriving the bound of the power of the electron number-density in (45). Let

$$\chi_R(\mathbf{x}) = 1, \text{ if } \mathbf{x} \text{ lies within a sphere of radius } R, \text{ and } = 0, \text{ otherwise.} \tag{46}$$

Then, clearly, for the probability to have the electrons within a sphere of radius R, we have

$$\begin{aligned} \mathrm{Prob}\big[\, |\mathbf{x}_1| \leq R, ..., |\mathbf{x}_N| \leq R \,\big] &\leq \mathrm{Prob}\big[\, |\mathbf{x}_1| \leq R \,\big] \\ &= \frac{1}{N} \int \mathrm{d}^3\mathbf{x}\, \chi_R(\mathbf{x})\, \varrho(\mathbf{x}) \\ &\leq \frac{1}{N} \Big[\int \mathrm{d}^3\mathbf{x}\, \varrho^{5/3}(\mathbf{x}) \Big]^{3/5} (v_R)^{2/5}, \end{aligned} \tag{47}$$

where in the last inequality, we use Hölder's inequality, the fact that $\chi_R(\mathbf{x})^{2/5} = \chi_R(\mathbf{x})$, and where $v_R = 4\pi R^3/3$.

From (45) and (47), we have the fundamental inequality (Manoukian and Sirininlakul, 2005)

$$\mathrm{Prob}\big[\, |\mathbf{x}_1| \leq R, ..., |\mathbf{x}_N| \leq R \,\big] \Big(\frac{N}{v_R}\Big)^{2/5} < 8\Big(\frac{1}{a_0^3}\Big)^{2/5} [1 + Z^{2/3}]^{6/5}, \tag{48}$$

where $a_0 = \hbar^2/me^2$ is the Bohr radius. One may infer from this equation the *inescapable* fact that *necessarily* for a non-vanishing probability of having the electrons within a sphere of radius R, the corresponding volume v_R grows not any slower than the first power of N for $N \to \infty$, *since otherwise the left-hand side of the inequality would go to infinity and would be in contradiction with the finite upper bound on the its right-hand side*. That is, *necessarily*, the radius R grows not any slower than $N^{1/3}$ for $N \to \infty$. No wonder matter occupies such a large volume!

5. Summary

We summarized the key results in the above analyses. The Pauli exclusion is not only sufficient for establishing that matter in the quantum setting is stable but is also necessary. This is precisely the condition that gives an N power law behavior of the ground-state energy, otherwise by revoking the exclusion principle, the ground-state energy would lead to power law behavior, N^α, $\alpha > 1$, (Lieb, 1979; Manonkian and Muthaporn, 2002; Maroukian and Muthaporn, 2003a; Muthaporn and Manoukian, 2004; Maroukian and Siriniolakul, 2004; Monoukian and Muthaporn, 2003b; Manoukian, 2013), implying instability as discussed in the Introduction. The fact that the electron density, as obtained from (44) satisfies the bound

$$\int \mathrm{d}^3\mathbf{x}\, \varrho^{5/3}(\mathbf{x}) < 32 \frac{m^2 \mathrm{e}^2}{\hbar^4} N [1 + Z^{2/3}]^2, \tag{49}$$

with an upper bound with a single power of N, which allows us to infer that matter with an extension radius R grows not any slower than $N^{1/3}$ for $N \to \infty$ as established in (48). "Bosonic matter" behaves completely differently and we refer the reader to the investigation in (Manoukian *et al*., 2006) for the relevant details involving its collapsing stage.

Acknowledgments

The authors would like to thank their colleagues at the Institute for their enthusiasm and the keen interest they have shown in this presentation. One of the authors (EBM) would also like to thank his colleagues C. Muthaporn and S. Sirininlakul for earlier discussions and collaborations.

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